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On Selberg's trace formula: chaos, resonances and time delays

Péter Lévay

Department of Theoretical Physics, Institute of Physics, Technical University of Budapest, H–1521 Budapest, Hungary

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Abstract. The quantization of the chaotic geodesic motion on Riemann surfaces $\Sigma_{g,\kappa}$ of constant negative curvature with genus g and a finite number of points κ infinitely far away (cusps) describing scattering channels is investigated. It is shown that terms in Selberg's trace formula describing scattering states can be expressed in terms of a renormalized time delay. This quantity is the time delay associated with the surface in question *minus* the time delay corresponding to the scattering problem on the Poincaré upper half-plane uniformizing our surface. Poles in these quantities give rise to resonances reflecting the chaos of the underlying classical dynamics. Our results are illustrated for the surfaces $\Sigma_{1,1}$ (Gutzwiller's leaky torus), $\Sigma_{0,3}$ (pants), and a class of $\Sigma_{g,2}$ surfaces. The generalization covering the inclusion of an *integer* $B \geqslant 2$ magnetic field is also presented. It is shown that the renormalized time delay is *not* dependent on the magnetic field. This shows that the semiclassical dynamics with an integer magnetic field is the same as the free dynamics.

1. Introduction

The quantization of the free motion of a particle on a surface of constant negative curvature has generated considerable interest over the past few years. From the classical point of view it provides a simple example of a dynamical system with hard chaos [1]. Moreover, according to the general philosophy of quantum chaos, in order to learn more about such systems it is instructive to investigate how this irregular behaviour manifests itself in the corresponding quantum system. It is now well known that the semiclassical trace formula of Gutzwiller [2] relating the classical periodic orbits to the quantum energy spectrum becomes exact and coincides with Selberg's trace formula [3] derived using pure group theoretical considerations. This striking coincidence has become the starting point of further investigations of Selberg's trace formula in order to establish the correspondence between the distribution of energy levels and the nontrivial zeros of the spectral zeta function expressed as a product over the periodic orbits [4]. As a next step, Comtet *et al* [5] observed that existing generalizations of the Selberg trace formula (well known to mathematicians) can be used to describe the physical situation of the motion of a particle on our surface in a constant magnetic field. These investigations, however, used compact-type surfaces only giving rise to bound states, or noncompact ones using special boundary conditions, hence 'killing out' the scattering states.

The description of scattering states and the corresponding S-matrices on noncompact surfaces was initiated by Faddeev [6], and developed in the book of Lax and Phillips [7]. The physical interpretation of these results in the context of quantum chaos was given by Gutzwiller [8, 9], the systematic adaptation of these ideas to surfaces with a multitude of cusps (i.e. points infinitely far away) regarding them as multichannel scattering systems is due to Pnueli [10].

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However, as far as physics is concerned these studies pay no attention to the role these quantities play in the generalization of Selberg's trace formula also valid for noncompact surfaces with a multitude of cusps. To the best of our knowledge, apart from a comment in the book of Gutzwiller [9] trying to interpret the sum appearing in the asymptotic behaviour of the scattering states as the analogue of the semiclassical trace formulae for positive energies, no work has tried to clarify such issues *based on the exising trace formula itself*. The aim of this paper is to fill this gap. Here we show that though the terms describing the scattering states in this formula seem formidable at first sight, they can be written in a nice form amenable to a natural physical interpretation. Our aim was also to present known mathematical results scattered throughout the literature in a form accessible to physicists.

The organization of this paper is as follows. In section 2 we describe our models of chaotic scattering. Here we review a number of results well known to mathematicians, which may sound, however, unfamiliar to physicists. Section 3 is devoted to a description of the scattering problem on the upper half-plane. Selberg's trace formula for noncompact finite-area surfaces with a multitude of cusps is introduced in section 4. It is rewritten in a nice form in terms of the Wigner–Smith time delays of the scattering problem on the surface and the upper half-plane uniformizing the surface. In section 5 we relate scattering resonances to the poles of the renormalized time delay. In section 6 we introduce the resolvent trace formula usually used in numerical calculations. Here we discuss the zeros of Selberg's zeta function and pay special attention to the zeros corresponding to scattering resonances. Here we also rewrite the functional equation for Selberg's zeta function in terms of the renormalized S-matrix. In section 7, by employing three particular models of chaotic scattering, the parabolic terms describing scattering states in the trace formula are examined. The first is Gutzwiller's leaky torus, the canonical example of chaotic scattering. The second is a surface with three cusps (pants). We also discuss a class of surfaces with two cusps corresponding to the choice of Hecke congruence groups $\Gamma_0(p)$, where p is a prime number of the form $p = 11 + 12k$, $k = 0, 1 \ldots$. In section 8 we consider the problem of quantization of the motion of a particle on our surface in a constant *integer* magnetic field $B \geqslant 2$. We show that the parabolic terms in the trace formula can again be expressed in terms of a renormalized time delay. However, now the time delay turns out to be independent of B. Our conclusions are left for section 9.

2. Chaotic scattering on Riemann surfaces

A large class of scattering systems exhibiting hard chaos can be obtained by considering the geodesic motion on *noncompact* Riemann surfaces $\Sigma_{g,\kappa,e}$ with finite area and constant negative Gaussian curvature. Here g denotes the genus (i.e. the number of 'holes') and κ stands for the number of cusps or leaks [8, 10] producing the scattering channels. The subscript e denotes the possible occurrence of the so-called elliptic points, their presence renders Σ to be an orbifold rather than a manifold. Though the terms corresponding to scattering states we are to describe here are not affected by such points, in the following, unless stated explicitly, we only consider surfaces without such points and assume that $2g + \kappa \geqslant 3$. The geodesic motion on such surfaces is known to be strictly ergodic and even strongly chaotic. From the mathematical point of view these surfaces can be obtained via Riemann uniformization which means the following. Take the Poincaré upper half-plane $\mathcal{H} \equiv \{z = x + \mathrm{i}y \in \mathbb{C} | y > 0\}$, with the Poincaré metric $\mathrm{d}s^2 = y^{-2}((\mathrm{d}x)^2 + (\mathrm{d}y)^2)$ of Gaussian curvature $K = -1$, and form the right coset $\Gamma\backslash\mathcal{H}$, where Γ is a Fuchsian group of the first kind acting on $\mathcal{H}$ discontinuously. Uniformization means that we represent our Riemann surface as this right coset viewed as a fundamental domain in $\mathcal{H}$ with its boundary points identified by elements of Γ, i.e. we have $\Sigma_{g,\kappa} \sim \Gamma\backslash\mathcal{H}$. The copies of the fundamental domain give a tessellation of the upper half-plane. Γ is a discrete subgroup of $PSL(2,\mathbb{R})$, the group of fractional linear transformations $\gamma = (ab|cd)$, $\gamma z = \frac{az+b}{cz+d}$, with the properties $a, b, c, d, \in \mathbb{R}$ and $ad - bc = 1$. Such discrete subgroups have a finite number of generators, and the noneuclidean area V of $\Sigma_{g,\kappa}$ is finite. Recall that using the Gauss–Bonnet theorem the area can be expressed as $V = 2\pi(2g - 2 + \kappa)$. The generators of Γ form the letters of an alphabet, and the elements of Γ are the possible words that can be combined from such letters. There are, however, some defining relations that can be used to simplify all possible combinations. Moreover, the triple (g, κ, e) is determined by the structure of the group [11]. The canonical example is the modular group $PSL(2,\mathbb{Z})$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, generated by the letters $U : z \to -\frac{1}{z}$ and $W : z \to z + 1$, with the triple $(0, 1, 2)$ and defining relations $U^2 = (UW)^3 = I$. An important class of Γ arises by considering the subgroup

$$\Gamma(N) \equiv \left\{ \gamma \in SL(2,\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod} N) \right\} \tag{1}$$

which is called the principal congruence group of level N. Any subgroup of the modular group which contains $\Gamma(N)$ is called a *congruence subgroup of level N*.

The quantum systems arising from the quantization of the geodesic motion on $\Sigma_{g,\kappa}$ are governed by Schrödinger's equation $H\psi = E\psi$, with the Hamiltonian $H = -\triangle$, where $\triangle = y^2(\partial_x^2 + \partial_y^2)$ is the Laplacian on $\mathcal{H}$ corresponding to the Poincaré metric, with $\psi(z)$ subject to the boundary condition $\psi(\gamma z) = \psi(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. (We set $\hbar = 2m = 1$ for convenience.) The spectrum of H is known to be both discrete and continuous [12]. Scattering solutions corresponding to the continuous spectrum with $E = \frac{1}{4} + k^2$, are described by the Eisenstein series which we now briefly review. (For physicists, a good reference on Eisenstein series can be found in [10], for the mathematically oriented reader we refer to [13].)

The transformations $\gamma \in \Gamma$ are called *parabolic* (*hyperbolic* and *elliptic*) if $|\operatorname{Tr}\gamma| = 2$, ($>2$, and <2, respectively). Such transformations can be shown to be conjugate to a translation $z \to z + a$, $a \in \mathbb{R}$ (dilatation $z \to bz$, $b > 0$, rotation, respectively). Among the generators of Γ there are κ parabolic ones $P_1, P_2, \ldots, P_\kappa$. The fixed points of these generators are the cusps. They will be denoted by $z_1, z_2, \ldots, z_\kappa$ and taken to be the elements of $\mathbb{R} \cup \infty$ (the boundary of $\mathcal{H}$) since they are infinitely far away with respect to the Poincaré metric. Under the identification of $\Gamma\backslash\mathcal{H}$ and $\Sigma_{g,\kappa}$ the cusps correspond to punctures ('leaks') of our surface describing scattering channels. For each $\alpha = 1, 2, \ldots \kappa$, the P_α generate an infinite cyclic subgroup Γ_α of Γ, the stability subgroup of cusp α. Since parabolic elements are conjugate to a translation we can choose an element of $\sigma_\alpha \in SL(2,\mathbb{R})$, such that $\sigma_\alpha\infty = z_\alpha$ and $\sigma_\alpha^{-1} P_\alpha \sigma_\alpha = (1 \pm 1|01)$. We denote by Γ_∞ the infinite cyclic group generated by W with its fixed point being ∞ the standard cusp. This is the group consisting of elements of the form $\pm(1b|01)$, $b \in \mathbb{Z}$. The Eisenstein series $\mathcal{E}_\alpha(z, s)$ corresponding to the cusp z_α of Γ is defined for $\operatorname{Re} s > 1$ by the absolutely convergent series

$$\mathcal{E}_\alpha(z, s) = \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} \operatorname{Im}(\sigma_\alpha^{-1}\gamma z)^s \qquad \alpha = 1, 2, \ldots, \kappa. \tag{2}$$

In this way the Eisenstein series defined satisfies the Schrödinger equation, i.e. $H\mathcal{E}_\alpha(z, s) = s(1 - s)\mathcal{E}_\alpha$ for each $\alpha = 1, 2, \ldots, \kappa$, and the boundary condition $\mathcal{E}_\alpha(\gamma z, s) = \mathcal{E}_\alpha(z, s)$ for all $\gamma \in \Gamma$. Of course, we are interested in the choice $s = \frac{1}{2} + \mathrm{i}k$ with $k \in \mathbb{R}$. For this purpose we need a meromorphic continuation of $E_\alpha(z, s)$ over the whole s-plane. This continuation exists and the poles of $E_\alpha(z, s)$ are all simple and in the segment $\frac{1}{2} < s \leqslant 1$. One can derive a Fourier expansion of $E_\alpha(z, s)$ at the cusp β which is of the form

$$\mathcal{E}_\alpha(\sigma_\beta z, s) = \delta_{\alpha\beta} y^s + \varphi_{\alpha\beta}(s) y^{1-s} + \sum_{n \neq 0} \varphi_{\alpha\beta}(n, s) W_s(nz) \tag{3}$$

where

$$\varphi_{\alpha\beta}(s) = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\sum_{c>0} c^{-2s}\mathcal{S}_{\alpha\beta}(0,0;c) \tag{4}$$

$$\varphi_{\alpha\beta}(n,s) = \frac{\pi^s}{\Gamma(s)}|n|^{s-1}\sum_{c>0} c^{-2s}\mathcal{S}_{\alpha\beta}(0,n;c). \tag{5}$$

Here $\mathcal{S}_{\alpha\beta}(m,n;c)$ are the so-called Kloosterman–Gauss–Ramanujan sums defined by

$$\mathcal{S}_{\alpha\beta}(m,n;c) = \sum_{\gamma=(a*|cd)\in\Gamma_\infty\backslash\sigma_\alpha^{-1}\Gamma\sigma_\beta/\Gamma_\infty} \mathrm{e}^{2\pi\mathrm{i}(ma+nd)/c} \tag{6}$$

and

$$W_s(z) = 2|y|^{1/2}K_{s-1/2}(2\pi|y|)\mathrm{e}^{2\pi\mathrm{i}x} \tag{7}$$

is Whittaker's function on $z\in\mathbb{C}\backslash\mathbb{R}$. Due to the asymptotic behaviour $W_s(z)\sim \mathrm{e}^{-2\pi y}$, as $y\to\infty$ the nonzero Fourier coefficients in equation (3) dye out exponentially when we approach any of our cusps. Hence only the term $\delta_{\alpha\beta}y^s+\varphi_{\alpha\beta}(s)y^{1-s}$ from equation (3) survives near the cusps. Since y^s and y^{1-s} correspond to the incoming and outgoing plane waves in hyperbolic geometry we are left with the correct asymptotic behaviour for scattering states. Moreover, from this it is clear that $\varphi_{\alpha\beta}(\frac{1}{2}+\mathrm{i}k)$ has to be proportional to the scattering matrix $S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}(k)$ of our surface $\Gamma\backslash\mathcal{H}$. Indeed, according to [7]

$$S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}(q,k) = -q^{-2\mathrm{i}k}\varphi_{\beta\alpha}(\tfrac{1}{2}+\mathrm{i}k) \tag{8}$$

where $0<q\in\mathbb{R}$ is arbitrary. For the mathematical proof of equation (8) we refer to [7]. In order to give some physical insight to its meaning we remark that the 'interaction' associated with our scattering problem is merely dictated by the geometrical properties of our surface. Hence we have no 'free dynamics', in the usual sense, with respect to which our system should be defined. However, we can proceed as follows [9]. We can put a ring (in hyperbolic geometry this is called a horocycle) on each cusp, regularizing the infinite length of the geodesic corresponding to the scattering trajectory coming from and then going to infinity through the leak. Using the σ_α transformations the neighbourhood of each cusp (cuspidal zone) can be mapped to the semi-strip $F_q\equiv\{z\in\mathcal{H}|y>q, 0<x<1\}$. Hence the value of q defines the horocycle which plays the role of a monitoring station; this is the place where the particle is registered after being scattered. This choice tells us where the 'free dynamics' start. The minus sign in equation (8) results in the nice property of $S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}(k)$ proved in proposition 8.14 of [7]:

$$S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}(0) = \delta_{\alpha\beta} \tag{9}$$

moreover, $S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}(k)$ is unitary and symmetric [7].

In order to know the scattering matrix of a particular surface the quantity $\varphi_{\alpha\beta}$ present in equation (4) has to be calculated. For this purpose, after fixing Γ uniformizing our surface, we have to describe the double cosets appearing in (6), and characterize our cusps—this is generally an arduous task. However, in spite of this we are provided with a variety of $\varphi_{\alpha\beta}$ matrices for different groups Γ having been calculated for different purposes by number theorists; see in particular, Hejhal's book [12]. These results were first communicated to physicists in [8–10, 14].

3. The scattering problem on the upper half-plane

We have seen that the space uniformizing our Riemann surfaces is the upper half-plane $\mathcal{H}$. Hence for later use it will be useful to describe scattering matrices associated with the Poincaré upper half-plane $\mathcal{H}$ itself. It is well known that $\mathcal{H}$ can be described as the Riemannian symmetric space $\mathcal{H} \sim SL(2,\mathbb{R})/SO(2)$. General symmetric spaces of noncompact type have already been widely used in algebraic scattering theory (AST), in order to describe quantum mechanical scattering problems characterized by a semisimple noncompact symmetry group G (see the paper of F Iachello in [15] and [16, 17]). In AST it is assumed that the Hamiltonian governing the scattering process is simply a function of the quadratic Casimir operator C of G. Then it is proved that the functional form of the S-matrix can be found (up to some arbitrary functions) via group theoretical manipulations alone, once a subgroup chain $G_1 \subset G_2 \subset \cdots G$ has been chosen. The remaining functions in S can be fixed by using a particular coordinate realization for G acting on some space. For $SL(2,\mathbb{R})$ acting on $\mathcal{H}$, the parabolic subgroup $E(1)$ generated by the infinitesimal operator $-\mathrm{i}\partial_x$ alone, plays a special role simply generating translations along the x direction of the upper half-plane. Hence we chose the subgroup chain as $E(1) \subset SL(2,\mathbb{R})$. Note that this is a noncanonical choice since the particular form $\mathcal{H} \sim SL(2,\mathbb{R})/SO(2)$ would rather imply the choice $SO(2) \subset SL(2,\mathbb{R})$. In [17] other nonstandard choices have been shown, and using the intertwining operator method the S-matrices have been calculated. For the subgroup chain $E(1) \subset SL(2,\mathbb{R})$ the functional form of the S-matrix turns out to be

$$S^{\mathcal{H}}(k) = |\lambda|^{-2\mathrm{i}k} c(k) \qquad |c(k)| = 1 \tag{10}$$

where $\lambda \in \mathbb{R}$ is the eigenvalue associated with the $E(1)$ subgroup. In order to specify the function $c(k)$, we chose the standard realization of the $SL(2,\mathbb{R})$ algebra, i.e. the infinitesimal generators of the $SL(2,\mathbb{R})$ action on $\mathcal{H}$ (see, e.g., [14]). The Casimir operator C for $SL(2,\mathbb{R})$ is just the Laplace–Beltrami operator $\triangle$ on $\mathcal{H}$, and the scattering Hamiltonian is identified as $-(C+\frac{1}{4})$, with eigenvalue k^2.

Hence on $\mathcal{H}$ we have to solve the equation $y^2(\partial_x^2+\partial_y^2)\Psi(x,y) = s(s-1)\Psi(x,y)$. Inserting $s = \frac{1}{2} + \mathrm{i}k$ then gives the scattering states. Of course, we also have to clarify the asymptotic behaviour of the solutions characterizing the scattering states. We seek solutions in the form $\Psi(x,y) = F(x)G(y)$, this yields, after the separation of variables, two ordinary differential equations,

$$\frac{G''}{G}(y) - s(s-1)y^{-2} = Q = -\frac{F''}{F}(x) \tag{11}$$

where Q is a separation constant. Solving for $F(x)$ gives $F(x) = \exp(\mathrm{i}\lambda x)$ with $Q = \lambda^2$. Setting $G(y) = y^{1/2}R(y)$ shows that $R(y)$ satisfies

$$y^2R'' + yR' - ((\lambda y)^2 + (s-\tfrac{1}{2})^2)R = 0. \tag{12}$$

If $\lambda = 0$ then $G(y) = y^s$, for $\lambda \neq 0$ $G(y) = y^{1/2}K_{s-1/2}(|\lambda|y)$. Using the functional equation $K_s(z) = K_{-s}(z)$ for $\mathrm{Re}\,(z) > 0$ and the asymptotics $K_s(z) \sim 2^{s-1}\Gamma(s)z^{-s}$ for $\mathrm{Re}\,s > 0$ and $\mathrm{Re}\,z > 0$ as $z \to 0$ of K-Bessel functions [18] we obtain the following formula for the asymptotic behaviour of $G(y)$:

$$G(y) \sim 2^{\mathrm{i}k}\Gamma(\mathrm{i}k)|\lambda|^{-\mathrm{i}k}y^{\frac{1}{2}-\mathrm{i}k} + 2^{-\mathrm{i}k}\Gamma(-\mathrm{i}k)|\lambda|^{\mathrm{i}k}y^{\frac{1}{2}+\mathrm{i}k} \qquad \text{as} \quad y \to 0^+. \tag{13}$$

From this we can read off the S-matrix:

$$S^{\mathcal{H}}(|\lambda|,k) = \left(\frac{|\lambda|}{2}\right)^{-2\mathrm{i}k} \frac{\Gamma(\mathrm{i}k)}{\Gamma(-\mathrm{i}k)} = -\left(\frac{|\lambda|}{2}\right)^{-2\mathrm{i}k} \frac{\Gamma(1+\mathrm{i}k)}{\Gamma(1-\mathrm{i}k)} \qquad \lambda \neq 0 \tag{14}$$

where we have used the formula $\Gamma(z+1) = z\Gamma(z)$. We can compare this result valid for $\mathcal{H}$ with the one obtained for $\Gamma\backslash\mathcal{H}$, see equations (4), (6) and (8). We can see that the *positive* numbers q and $|\lambda|$ play a similar role by acting as scaling parameters in the corresponding S-matrices.

Before concluding this section we remark that by using the substitution $y(r) = \mathrm{e}^{-r}$, equation (12) can be written as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \lambda^2 \mathrm{e}^{-2r} + k^2\right) R(r) = 0 \tag{15}$$

which is just a Schrödinger equation in one dimension with a potential $V(r) = \lambda^2 \mathrm{e}^{-2r}$. Hence our S-matrix of equation (14) is the scattering matrix for this problem.

4. Selberg's trace formula and the renormalized time delay

Now we are ready to introduce Selberg's trace formula valid for noncompact surfaces having finite area. Let $h(k)$ be a function satisfying the following conditions:

$$\begin{array}{ll} h(k) & \text{is even} \\ h(k) & \text{holomorphic in the strip} \quad |\mathrm{Im}\, k| \leqslant \frac{1}{2} + \varepsilon \\ h(k) \ll (|k|+1)^{-2-\varepsilon} & \text{in the strip.} \end{array} \tag{16}$$

In order to motivate the physical meaning of $h(k)$, we note that the special choice $h(k) = \mathrm{e}^{-(1/4+k^2)} = \mathrm{e}^{-Et}$ enables one to study the properties of the heat kernel $\mathrm{e}^{t\triangle}$ of $\triangle$. Let, moreover, $g(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}uk} h(k)\,\mathrm{d}k$ be the Fourier transform of $h(k)$. Then, Selberg's trace formula for noncompact surfaces without elliptic points is [12]

$$\begin{aligned} \sum_j h(k_j) = {} & \frac{V}{4\pi}\int_{-\infty}^{+\infty} k h(k)\tanh(\pi k)\,\mathrm{d}k + \sum_{\mathrm{PPO}}\sum_{n=1}^{\infty} \frac{l(p)}{2\sinh(nl(p)/2)} g(nl(p)) \\ & + \frac{1}{4} h(0)\,\mathrm{Tr}\left(I - \Phi\left(\frac{1}{2}\right)\right) - \kappa g(0)\log 2 - \frac{\kappa}{2\pi}\int_{-\infty}^{\infty} h(k)\psi(1+\mathrm{i}k)\,\mathrm{d}k \\ & + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(k)\,\mathrm{Tr}\left(\Phi'\left(\frac{1}{2}+\mathrm{i}k\right)\Phi\left(\frac{1}{2}+\mathrm{i}k\right)^{-1}\right)\mathrm{d}k. \end{aligned} \tag{17}$$

Here $V = 2\pi(2g - 2 + \kappa)$ is the area of the surface, Φ is the $\kappa \times \kappa$ matrix with entries $\varphi_{\alpha\beta}$ as given by (4), $\psi(z) \equiv \frac{\Gamma'}{\Gamma}(z)$ is the digamma function [18], I is the $\kappa \times \kappa$ identity matrix. $E_j \equiv \frac{1}{4} + k_j^2$ are the energy values belonging to the discrete part of the spectrum. The first two terms on the right-hand side of the formula are well known from studies concerning the compact case [9]. The first is the so-called Weyl term, and the second contains the sum over the primitive periodic orbits (PPO) of primitive length $l(p)$. The repetitions of the orbits are indexed by n. The remaining four terms, which are our main concern here, correspond to the modification of the trace formula due to the presence of scattering states. In the following we shall refer to these terms as the *parabolic contribution* to the trace formula.

Our aim is now to rewrite the parabolic contribution in a physically more transparent form. Indeed, during algebraic manipulations found in the mathematical literature the physical origin and meaning of these terms is by no means clear. As a first step, using the fact that $h(k)$ is even, we rewrite the second and third terms of this contribution in the form

$$\begin{aligned} & -\frac{\kappa}{4\pi}\int_{-\infty}^{\infty} h(k)(2\log 2 + \psi(1+\mathrm{i}k) + \psi(1-\mathrm{i}k))\,\mathrm{d}k \\ & \qquad = +\mathrm{i}\frac{\kappa}{4\pi}\int_{-\infty}^{\infty} h(k)\partial_k \log\left(2^{2\mathrm{i}k}\frac{\Gamma(1+\mathrm{i}k)}{\Gamma(1-\mathrm{i}k)}\right)\mathrm{d}k. \end{aligned} \tag{18}$$

Similarly, for the fourth term using the identity $\mathrm{Tr}\log M = \log \mathrm{Det} M$, we get

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(k)\,\mathrm{Tr}\left(\Phi'\left(\frac{1}{2}+\mathrm{i}k\right)\Phi\left(\frac{1}{2}+\mathrm{i}k\right)^{-1}\right)\mathrm{d}k$$

$$= -\mathrm{i}\frac{1}{4\pi}\int_{-\infty}^{\infty} h(k)\partial_k \log \mathrm{Det}\left(\Phi\left(\frac{1}{2}+\mathrm{i}k\right)\right) \mathrm{d}k. \tag{19}$$

Let us now define the $\kappa \times \kappa$ matrix using equation (14):

$$S^{\mathcal{H}}_{\alpha\beta}(|\lambda|, k) \equiv S^{\mathcal{H}}(|\lambda|, k)\delta_{\alpha\beta}. \tag{20}$$

Moreover, recalling equation (8), we then have two matrices $S^{\mathcal{H}}_{\alpha\beta}(|\lambda|, k)$ and $S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}(q, k)$ which are also dependent on the *positive* quantities q and $|\lambda|$. In addition, we know that with the help of q we can fix where the free dynamics starts. Let us refer our dynamics on $\Gamma\backslash\mathcal{H}$ to the dynamics on $\mathcal{H}$ by giving q and $|\lambda|$ *the same values*. With this convention the quantity

$$-\mathrm{i}\frac{1}{4\pi}\int_{-\infty}^{\infty} h(k)\partial_k \log \mathrm{Det}\boldsymbol{S}(k)\, \mathrm{d}k \qquad \boldsymbol{S}(k) \equiv S^{\Gamma\backslash\mathcal{H}}(k)(S^{\mathcal{H}}(k))^{-1} \tag{21}$$

is independent of $q = |\lambda|$ and equals the last three terms of our parabolic contribution. Moreover, by virtue of equations (8) and (9) the first term from the parabolic contribution is $\frac{\kappa}{2}h(0)$. Since scattering matrices always occur in the combination as shown in (21), in the following we shall refer to them as $S^{\Gamma\backslash\mathcal{H}}(k)$ and $S^{\mathcal{H}}(k)$, i.e. as the ones independent of q and $|\lambda|$. Moreover, we call $\boldsymbol{S}(k)$ the renormalized S-matrix.

Introducing the Wigner–Smith time delay [19] for the corresponding S-matrices as

$$T^{\Gamma\backslash\mathcal{H}}(k) \equiv \frac{\mathrm{i}}{2k}\partial_k \log \mathrm{Det} S^{\Gamma\backslash\mathcal{H}}(k) \tag{22}$$

(and similarly for $S^{\mathcal{H}}(k)$) we can finally write the parabolic contribution ($\mathrm{d}E = 2k\,\mathrm{d}k$) in the nice form

$$\frac{\kappa}{2}h(0) - \frac{1}{2\pi}\int_0^{\infty} h(E)T(E)\,\mathrm{d}E \qquad \text{where} \quad T(E) \equiv T^{\Gamma\backslash\mathcal{H}}(E) - T^{\mathcal{H}}(E) \tag{23}$$

which is the main result of the paper. We shall refer to $T(E)$ as the *renormalized time delay*. This quantity is the time delay associated with the surface in question *minus* the time delay corresponding to the scattering problem on the Poincaré upper half-plane uniformizing our surface.

We now make a few important comments. Intuitively, the time delay can be imagined as the difference between the time spent by the scattered particle within the region of interaction and the time that it would have spent in the same region had it moved freely. In our case the free dynamics is the one on the Poincaré upper half-plane $\mathcal{H}$. Here, the particle trajectory is determined by the intrinsic geometry of $\mathcal{H}$. When quantizing this free (geodesic) motion we have to solve Schrödinger's equation $H\psi = E\psi$, where H represents $-\triangle$. When replacing the free dynamics by the interacting one, the role of interaction is not played by an interaction term (potential) but by the *special boundary condition* we impose on the free system. When quantizing, this interaction manifests itself via the condition $\psi(gz) = \psi(z)$ $g \in \Gamma$ we impose on the wavefunction. Now interaction is just restriction of the motion to the fundamental domain $\Gamma\backslash\mathcal{H}$ in $\mathcal{H}$. Identifying the free and interacting dynamics in this way, being a time difference we can alternatively regard the renormalized time delay of equation (23) as the *time delay* for a scattering problem which is purely geometric in origin.

Turning back to our trace formula first, we chose $h(k) = \mathrm{e}^{-(1/4+k^2)t} = \mathrm{e}^{-Et}$. Clearly, this function satisfies the conditions of equation (16). Then the left-hand side of the trace formula is simply $\mathrm{Tr}\,\mathrm{e}^{t\triangle}$, i.e. the trace of the heat kernel of $\triangle$. In this case it is easy to show that $g(u) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{t/4-u^2/4t}$, which is to be used in the second and fourth terms on the right-hand side of equation (17). Now equation (23) shows that up to the term $\frac{\kappa}{2}h(0)$ the parabolic contribution taking into account the scattering states is just a Laplace integral of the renormalized time delay $T(E)$. Note that the term $\sum_j h(k_j)$ on the left-hand side of equation (17) can also be written as a Laplace integral,

$$\sum_j \mathrm{e}^{-E_j t} = \int_0^\infty \mathrm{e}^{-Et}\rho(E)\,\mathrm{d}E \qquad \text{with} \quad \rho(E) = \frac{\mathrm{d}N(E)}{\mathrm{d}E} = \delta\sum_j (E - E_j) \tag{24}$$

where $N(E)$ is the number of eigenstates with energy $\leqslant E$, and $\rho(E)$ is the density of states for the discrete part of the spectrum. Looking at expression (23) we see that the term $\frac{1}{2\pi}\int_0^\infty \mathrm{e}^{-Et}T^{\Gamma\backslash\mathcal{H}}(E)\,\mathrm{d}E$ plays a similar role for the continuous part of the spectrum. With this observation the trace formula takes the form

$$\int_0^\infty \mathrm{e}^{-Et}\rho(E)\,\mathrm{d}E + \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{-Et}T(E)\,\mathrm{d}E - f_0(t) = \sum_{\mathrm{PPO}}\sum_{n=1}^{\infty}\frac{l(p)}{2\sinh(nl(p)/2)}g(nl(p)) \tag{25}$$

where

$$\begin{aligned} f_0(t) &\equiv \frac{V}{4\pi}\int_0^\infty \mathrm{e}^{-Et}\tanh(\pi\sqrt{E})\,\mathrm{d}E + \frac{\kappa}{2}\mathrm{e}^{-t/4} \\ g(nl(p)) &= \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{t/4-(nl(p))^2/4t}. \end{aligned} \tag{26}$$

This equation represents the response of our quantum system at the time $t > 0$ for a perturbation from the outside coming as a sharp blow at time $t = 0$. Note that the left-hand side of this formula contains the quantum mechanical quantities $\rho(E)$ and $T(E)$, while the right-hand side is purely classical.

Moreover, recall that the quantity $N_{\Gamma\backslash\mathcal{H}}(E) = \int_0^E \rho(E')\,\mathrm{d}E'$ is just the number of eigenvalues of $-\triangle$ on $\Gamma\backslash\mathcal{H}$ in the discrete part of the spectrum less then E. Similarly, we can define the quantity

$$M_{\Gamma\backslash\mathcal{H}}(E) = \frac{1}{2\pi}\int_0^E T^{\Gamma\backslash\mathcal{H}}(E')\,\mathrm{d}E' \tag{27}$$

to play the role of a counting function for the continuous spectrum. It is just the integral of Wigner's time delay calculated for the scattering problem on $\Gamma\backslash\mathcal{H}$ from 0 to the value E. The asymptotic behaviour of the sum $N_{\Gamma\backslash\mathcal{H}}(E) + M_{\Gamma\backslash\mathcal{H}}(E)$ is Weyl's law, its strongest form for a general surface (see [21] and references therein) is

$$N_{\Gamma\backslash\mathcal{H}}(E) + M_{\Gamma\backslash\mathcal{H}}(E) = \frac{V}{4\pi}E - \frac{\kappa}{\pi}\sqrt{E}\log\sqrt{E} + c_\Gamma\sqrt{E} + \mathrm{O}\left(\frac{\sqrt{E}}{\log\sqrt{E}}\right) \qquad \text{as} \quad E \to \infty \tag{28}$$

with c_Γ is a constant. It remains an open question as to which part of the spectrum is larger. For the special case of congruence groups Selberg has proved that $M_{\Gamma\backslash\mathcal{H}}(E)$ is much smaller than $N_{\Gamma\backslash\mathcal{H}}(E)$, and there are indications that the opposite situation for generic groups is more likely to occur [21].

5. Resonances

The renormalized time delay introduced in the previous section is a function of the scattering energy. It is a measure of the time spent by the particle in the leaky box. It can occur that for special values of the energy the particle is captured for a much longer period of time. For such values we have *resonance*.

The $T^{\mathcal{H}}(k)$ part of the renormalized time delay is [18]

$$T^{\mathcal{H}}(k) = -\frac{\kappa}{k}(\log 2 + \mathrm{Re}\,\psi(1+\mathrm{i}k)) = -\frac{\kappa}{k}\left(\log 2 - \gamma + k^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+k^2)}\right) \tag{29}$$

where $\gamma = 0.577\,21\ldots$ is Euler's constant. Moreover, since $\psi(1) = -\gamma$ we have $\lim_{k\to 0} kT^{\mathcal{H}}(k) = \kappa(\gamma - \log 2)$, and $\lim_{k\to\infty} kT^{\mathcal{H}}(k) \simeq -\log 2k$. Hence $-kT^{\mathcal{H}}(k)$ is a slowly varying function of k increasing monotonically from $\kappa(\log 2 - \gamma) > 0$ with an asymptotic $\log 2k$ behaviour. Hence $T^{\mathcal{H}}(k)$ merely gives a slowly varying smooth contribution to $T(k) = T^{\Gamma\backslash\mathcal{H}}(k) - T^{\mathcal{H}}(k)$.

In order to find resonances we have to investigate the pole structure of the $T^{\Gamma\backslash\mathcal{H}}(s)$ part of $T(s)$ as a function of the *complex* variable. To do this we first note that the matrix $\varphi_{\alpha\beta}(s)$ as given by (4) is just a Dirichlet series for $\mathrm{Re}\,s > 1$. Hence its determinant needed for the (22) time delay $T^{\Gamma\backslash\mathcal{H}}(s) \sim \frac{1}{2s-1}\partial_s \log \mathrm{Det}\Phi(s)$ is also given by a Dirichlet series, i.e.

$$\mathrm{Det}\Phi(s) = \left(\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\right)^{\kappa}\sum_{n=1}^{\infty} a_n g_n^{-2s} \tag{30}$$

with $a_1 \neq 0$ and $0 < g_1 < g_2 < \cdots < g_n \to +\infty$. It is known [12] that for $\mathrm{Re}\,s \geqslant \frac{1}{2}$, $\mathrm{Det}\Phi(s)$ has a *finite* number of poles, $s_a = \varrho_a$, $a = 0, 1, 2\ldots M$, all in the interval $\frac{1}{2} < s_a \leqslant 1$. Moreover, we also have the relation $\mathrm{Det}\Phi(s)\mathrm{Det}\Phi(1-s) = I$. These poles give rise to the eigenvalues $0 \leqslant E_a = s_a(1-s_a) < \frac{1}{4}$ in the so-called *residual spectrum*. The value $E_0 = 0$ with the value $s_0 = 1$ corresponds to the constant normalized solution $\Psi_0 \equiv V^{-1/2}$ of the Hamiltonian $H = -\Delta$. For $\mathrm{Re}\,s < \frac{1}{2}$ the poles are denoted by $s_\mu = \varrho_\mu + \mathrm{i}\eta_\mu$, $\mu = 1, 2, \ldots$. Then, we have the formula [12]

$$-\partial_s \log \mathrm{Det}\Phi(s) = \sum_j\left(\frac{1}{s-s_j} - \frac{1}{s-1+s_j^*}\right) + 2\log g_1 \tag{31}$$

where the sum for j is over $a = 1, 2\ldots M$, and $\mu = 1, 2, \ldots$. Hence on the critical line $s = \frac{1}{2} + \mathrm{i}k$ for $T^{\Gamma\backslash\mathcal{H}}(k)$ we have

$$T^{\Gamma\backslash\mathcal{H}}(k) = \frac{1}{2k}\left(\sum_\mu \frac{1-2\varrho_\mu}{(\frac{1}{2}-\varrho_\mu)^2 + (k-\eta_\mu)^2} + \sum_{a=0}^{M}\frac{1-2\varrho_a}{(\frac{1}{2}-\varrho_a)^2 + k^2} + 2\log g_1\right). \tag{32}$$

Note that the first sum on the right-hand side of this formula is *positive* and the second (corresponding to the possible presence of the residual spectrum) is *negative*. An important theorem states that for congruence groups we have no residual spectrum besides the obvious point $s_0 = 1$ [21]. Hence in this case the second sum merely gives the term $-\frac{1}{2k}\frac{1}{1/4+k^2}$. Terms coming from the first sum with $\mathrm{Re}\,s_\mu < \frac{1}{2}$ give rise to poles corresponding to resonances. Using equations (27) and (32), an easy application of Cauchy's theorem shows that the counting function $M_{\Gamma\backslash\mathcal{H}}(E)$ is approximately equal to the number of complex poles (resonances) with imaginary part less then E, on the left of the critical line $s = \frac{1}{2} + \mathrm{i}\sqrt{E}$. The distribution of these poles shows the irregular behaviour of the quantum scattering problem, hence reflecting the chaotic nature of the associated classical dynamics.

Since $T^{\mathcal{H}}(k)$ merely gives a slowly varying smooth contribution to $T(k) = T^{\Gamma\backslash\mathcal{H}}(k) - T^{\mathcal{H}}(k)$, the expression for $T(k)$ is dominated by terms of the form

$$T(k) \sim \frac{1}{k}\sum_\mu \frac{\Gamma_\mu/2}{(k-k_\mu)^2 + (\Gamma_\mu/2)^2} \tag{33}$$

which consist of a collection of Lorentzians centred at $k_\mu \equiv \eta_\mu$, with a half width $\Gamma_\mu/2 = \frac{1}{2} - \varrho_\mu$. The quantity $((\frac{1}{2}-\varrho_\mu)\eta_\mu)^{-1}$ can be thought of as a resonance lifetime. The allowed values for the quantities k_μ and Γ_μ are determined by the number theoretic properties of the Dirichlet series appearing in the determinant of the S-matrix. In turn, these properties can be traced back to the behaviour of the Kloosterman sums in (4). Note that when the distance between adjacent values of k_μ becomes less than Γ_μ the resonances start to overlap. This shows that the statistical properties of the resonances have to be investigated by the general methods developed by Fyodorov and Sommers [22] for studying the chaotic behaviour of quantum scattering.

6. The resolvent trace formula

In order to use the trace formula of equation (17) to relate the quantal data to the classical, we choose a special function $h(k)$ and exploit the pole structure of the renormalized time delay. Let us choose h and g as follows:

$$\begin{aligned} h_{s,\sigma}(k) &= ((s-\tfrac{1}{2})^2+k^2)^{-1} - ((\sigma-\tfrac{1}{2}2)^2+k^2)^{-1} \\ g_{s,\sigma}(u) &= \frac{\mathrm{e}^{-|u|(s-1/2)}}{2s-1} - \frac{\mathrm{e}^{-|u|(\sigma-1/2)}}{2\sigma-1}. \end{aligned} \tag{34}$$

Here $\sigma > \mathrm{Re}\, s > 1$, and σ is the regulator. This constraint is sufficient for ensuring the conditions listed in (16) for $h(k)$. In this case the final formula, called the resolvent trace formula, can be written as

$$\sum_j \left(\frac{1}{(s-1/2)^2+k_j^2} - \frac{1}{(\sigma-1/2)^2+k_j^2} \right) = \mathcal{F}(s) - \mathcal{F}(\sigma) \tag{35}$$

where

$$\mathcal{F}(s) = -\frac{V}{2\pi}\psi(s) + \frac{2}{2s-1}\sum_{\mathrm{PPO}}\sum_{m=0}^{\infty}\frac{l(p)}{\mathrm{e}^{(s+m)l(p)}-1} + \frac{2\kappa}{(2s-1)^2} - \frac{1}{2\pi}\int_0^\infty h_s(E)T(E)\,\mathrm{d}E \tag{36}$$

where $h_s(E) \equiv (s(s-1)+E)^{-1}$, and $T(E)$ are given by equations (22) and (23). For the derivation of the first two terms of this formula we refer to the paper of McKean [20]. In order to conform with the usual notation used by physicists for the Weyl term, we note that $\psi(s) - \psi(\sigma) = \sum_{n=0}^{\infty}(\sigma+n)^{-1} - (s+n)^{-1}$ [18]. The second term on the right-hand side of equation (36) represents the sum over classical periodic orbits well known from studies of quantum chaos on compact surfaces. Introducing Selberg's zeta function

$$Z(s) = \prod_{\mathrm{PPO}}\prod_{m=0}^{\infty}(1-\mathrm{e}^{-(s+m)l(p)}) \qquad \mathrm{Re}\,(s) > 1 \tag{37}$$

this sum over the periodic orbits can be written with the help of the logarithmic derivative of $Z(s)$ as

$$\frac{1}{2s-1}\frac{Z'}{Z}(s) = \frac{1}{2s-1}\sum_{\mathrm{PPO}}\sum_{m=0}^{\infty}\frac{l(p)}{\mathrm{e}^{(s+m)l(p)}-1}. \tag{38}$$

Substituting this into equation (36), we expect that the resolvent trace formula will give the analytic continuation of $\frac{Z'}{Z}(s)$ to the whole complex plane. In order to fulfil our expectations we have to clarify the pole structure of the fourth term in equation (36), which is the main contribution corresponding to the scattering states. Since we are only interested in the *pole structure*, in the following we only work up to s-independent terms. First, we evaluate the integral

$$\frac{1}{2\pi}\int_0^\infty h_{s,\sigma}(E)T^{\mathcal{H}}(E)\,\mathrm{d}E = -\frac{\kappa}{2\pi}\int_{-\infty}^{\infty} h_{s,\sigma}(k)(\log 2 + \psi(1+\mathrm{i}k))\,\mathrm{d}k. \tag{39}$$

Using the formula [18]

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{(z+n)}\right) \tag{40}$$

and the residue theorem for a suitable contour C, one has to evaluate the integral $-\frac{\kappa}{2\pi \mathrm{i}}\int_C h_{s,\sigma}(z)\psi(1+z)\,\mathrm{d}z$. The result is

$$-\frac{1}{2\pi}\int_0^{\infty} h_s(E)T^{\mathcal{H}}(E)\,\mathrm{d}E = \kappa\left(\sum_{n=1}^{\infty}\frac{1}{(s-1/2)^2 - n^2} - \frac{\psi(3/2-s)}{2s-1}\right) + \cdots \tag{41}$$

where the dots refers to the s-independent terms.

Our next task is to present a similar calculation for the quantity

$$-\frac{1}{2\pi}\int_0^{\infty} h_{s,\sigma}(E)T^{\Gamma\backslash\mathcal{H}}(E)\,\mathrm{d}E = -\mathrm{i}\frac{1}{4\pi}\int_{-\infty}^{\infty} h_{s,\sigma}(k)\partial_k \log \mathrm{Det}\left(\Phi\left(\frac{1}{2}+\mathrm{i}k\right)\right)\mathrm{d}k \tag{42}$$

where Φ is the $\kappa\times\kappa$ matrix with entries $\varphi_{\alpha\beta}$ as given by (4) and related to the scattering matrix by (8).

Now one has to evaluate the integral $\frac{1}{4\pi \mathrm{i}}\int_{C_R} h_{s,\sigma}(z)\partial_z \log \mathrm{Det}(\Phi(z))\,\mathrm{d}z$, where C_R consists of a line starting at the point $\frac{1}{2}-\mathrm{i}R$ and ending at $\frac{1}{2}+\mathrm{i}R$; we close the contour with a half-circle of radius R going through the point $\frac{1}{2}-R$. Here we remind the reader that $h_{s,\sigma}(z) = [(s-\frac{1}{2})^2 - (z-\frac{1}{2})^2]^{-1} - [(\sigma-\frac{1}{2})^2 - (z-\frac{1}{2})^2]^{-1}$ which has poles at $z = s$ and $z = 1-s$. The quantity $\partial_s \log \mathrm{Det}\Phi(s)$ for $\mathrm{Re}\, z < \frac{1}{2}$ has the poles z_μ, $\mu = 1, 2, \ldots,$ in the integration contour C_R. They give a contribution to the logarithmic derivative with *negative* residue. The poles z_a, $a = 0, 1, \ldots M$, located at the interval $\frac{1}{2} < \mathrm{Re}\, z \leqslant 1$ are not contained in C_R. However, due to the functional equation [12] $\mathrm{Det}\Phi(s)\mathrm{Det}\Phi(1-s) = 1$ the poles of $\mathrm{Det}\Phi(s)$ in $\frac{1}{2} < \mathrm{Re}\, z \leqslant 1$ are the *zeros* of $\mathrm{Det}\Phi(1-s)$ in $0 \leqslant \mathrm{Re}\, z < \frac{1}{2}$, hence they give a contribution to the logarithmic derivative with *positive* residue. The contribution from the half-circle in the limit $R\to\infty$ is zero due to the pole structure of $\partial_s \log \mathrm{Det}\Phi(s)$ and the properties of the function $h_{s,\sigma}$. Now, using the residue theorem we get the following result:

$$\begin{aligned}
-\frac{1}{2\pi}\int_0^{\infty} & h_{s,\sigma}(E)T^{\Gamma\backslash\mathcal{H}}(E)\,\mathrm{d}E \\
&= \frac{1}{2}\frac{1}{2s-1}\partial_s \log \mathrm{Det}\Phi(s) + \frac{1}{2}\sum_{a=0}^{M}\frac{1}{(s-1/2)^2 - (z_a-1/2)^2} \\
&\quad -\frac{1}{2}\sum_{\mu}\left(\frac{1}{(s-1/2)^2-(z_\mu-1/2)^2} - \frac{1}{(\sigma-1/2)^2-(z_\mu-1/2)^2}\right) + \cdots.
\end{aligned} \tag{43}$$

Here, as usual, the dots indicate the presence of s-independent terms. Note that $\frac{1}{2s-1}\partial_s \log \mathrm{Det}\Phi(s)$ is just the time delay analytically continued to the whole complex plane s. Moreover, since the μ sum is infinite we also included the $\sigma > 1$ term, rendering our sum absolutely convergent.

Now we have every term to characterize the poles and zeros of Selberg's zeta function $Z(s)$. Indeed, collecting everything up to s-independent terms we have

$$\begin{aligned}
\frac{1}{s-1/2}\frac{Z'}{Z}(s) &= \frac{V}{\pi}(\psi(s)-\psi(\sigma)) + \mathcal{T}(s) + \sum_j\left(\frac{1}{(s-1/2)^2+k_j^2} - \frac{1}{(\sigma-1/2)^2+k_j^2}\right) \\
&\quad + \sum_{\mu}\left(\frac{1}{(s-1/2)^2-(z_\mu-1/2)^2} - \frac{1}{(\sigma-1/2)^2-(z_\mu-1/2)^2}\right)
\end{aligned}$$

$$-\sum_{a=0}^{M}\frac{1}{(s-1/2)^2-(z_a-1/2)^2}-2\kappa\sum_{n=1}^{\infty}\frac{1}{(s-1/2)^2-n^2}$$
$$-\kappa\frac{1}{(s-1/2)^2}+\cdots \tag{44}$$

$$\mathcal{T}(s)\equiv\frac{1}{2s-1}\left(2\kappa\left(\psi\left(\frac{3}{2}-s\right)+\log 2\right)-\partial_s\log\mathrm{Det}\Phi(s)\right). \tag{45}$$

Using the well known properties of the logarithmic derivative of a complex meromorphic function one can easily see that the nontrivial zeros of $Z(s)$ are as follows. (a) They are on the line $\mathrm{Re}\, s=\frac{1}{2}$ localized symmetrically with respect to the real axis, or $s_j\in[0,1]$ symmetrically with respect to $s=\frac{1}{2}$. They correspond to the eigenvalues of H, $E_j=s_j(1-s_j)$ of the form $s_j=\frac{1}{2}+\mathrm{i}k_j$, corresponding to the discrete part of the spectrum. The multiplicity of the zeros equals the multiplicity of E_j. (b) They are at the points $s_\mu\equiv z_\mu$ which correspond to the poles of $\mathrm{Det}\Phi(s)$, i.e. the poles of the determinant of the scattering matrix on $\Gamma\backslash\mathcal{H}$ with the property $\mathrm{Re}\, s<\frac{1}{2}$. The multiplicity is no larger than κ, i.e. the number of scattering channels. These zeros correspond to the scattering resonances and are our main concern here. (c) Using formula (45) we can see that there are zeros in $Z(s)$ coming from the poles of $-\partial_s\log\mathrm{Det}\Phi(s)$, $s_a=1-z_a$, and $s_a=z_a$, $a=0,1,2\ldots M$, corresponding to the residual spectrum. If we have no residual spectrum (e.g. for congruence groups) we merely have the obvious point $z_0=1$. (d) There are also trivial zeros coming from the Weyl term. These are at the points $s=-n$, $n=0,1,2\ldots$, with multiplicity $\frac{A}{\pi}(n+\frac{1}{2})$. The remaining terms from (44) give rise to *poles* of $Z(s)$.

Hence we see that $Z(s)$, which is a quantity expressed in terms of the *classical* data (i.e. the length spectra of the periodic orbits), determines the *quantum* data, namely the eigenvalues and the scattering resonances. From the numerical point of view this amounts to finding the *zeros* of $Z(s)$.

Evaluating equation (44) at the points s and $1-s$ we obtain the equation

$$\frac{Z'}{Z}(s)+\frac{Z'}{Z}(1-s)=V\left(s-\frac{1}{2}\right)\tan\pi\left(s-\frac{1}{2}\right)+\left(s-\frac{1}{2}\right)(\mathcal{T}(s)-\mathcal{T}(1-s)) \tag{46}$$

where we have used the fact that

$$\pi\tan\pi\left(s-\frac{1}{2}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{1-s+n}-\frac{1}{s+n}\right). \tag{47}$$

Using the property $\log\mathrm{Det}\Phi(s)=\log\mathrm{Det}\Phi(1-s)$, we see that the final result can be expressed in terms of the *renormalized* S-matrix $\boldsymbol{S}(s)$ analytically continued to the whole complex s-plane in the following form:

$$\partial_s(\log Z(s)-\log Z(1-s))=V(s-\tfrac{1}{2})\tan\pi(s-\tfrac{1}{2})-\partial_s\log\mathrm{Det}\boldsymbol{S}(s). \tag{48}$$

Integrating this we obtain the nice form for the functional equation for Selberg's zeta function,

$$Z(1-s)=\frac{\mathrm{Det}\boldsymbol{S}(s)}{\mathrm{Det}\boldsymbol{S}(1/2)}Z(s)\mathrm{e}^{-V\int_0^{s-1/2}x\tan\pi x\,\mathrm{d}x}. \tag{49}$$

Moreover, according to [7] (see also (9)) the quantity $\mathrm{Det}\boldsymbol{S}(\frac{1}{2})$ (arising as an integration constant) can be fixed to one.

7. Examples

Now we consider some examples to illustrate the general formalism outlined above. Our first example is Gutzwiller's leaky torus which represents the best studied example of chaotic quantum scattering [8–10, 14]. In this case the Fuchsian group Γ is generated by two letters, $A = \left(\begin{smallmatrix}1 & 1\\ 1 & 2\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}1 & -1\\ -1 & 2\end{smallmatrix}\right)$, satisfying the defining relations $B^{-1}A^{-1}BA = -W^6$. (Recall that W is the transformation $z \mapsto z+1$, see the beginning of section 2.) This choice gives a Riemann surface of type $(g, \kappa, e) = (1, 1, 0)$, i.e. topologically a torus with one point infinitely far away. In this case the matrix $\varphi_{\alpha\beta}$ in (3) is a one-by-one unitary matrix, i.e. a phase factor. The description of the double cosets in (6) is very simple, and it turns out that the Kloosterman sum $\mathcal{S}(0, 0; c)$ is just Euler's function, i.e. the number of $d(\bmod c)$ relatively prime to c. Using this result one then proves that

$$\sum_{c=1}^{\infty} c^{-2s}\mathcal{S}(0, 0; c) = \frac{\zeta(2s-1)}{\zeta(2s)}. \tag{50}$$

Using this identity in equation (4) with $s = \frac{1}{2} + \mathrm{i}k$ and the functional relation $Z(s) = Z(1-s)$ with $Z(s) = \Gamma(s/2)\zeta(s)\pi^{-s/2}$ satisfied by the Riemann-zeta function, the scattering matrix can be given the following form:

$$S^{\Gamma\backslash\mathcal{H}}(q, k) = -\left(\frac{q}{\pi}\right)^{-2\mathrm{i}k} \frac{\Gamma(1/2 - \mathrm{i}k)}{\Gamma(1/2 + \mathrm{i}k)} \frac{\zeta(1 - 2\mathrm{i}k)}{\zeta(1 + 2\mathrm{i}k)}. \tag{51}$$

Now recall equation (14) with $|\lambda| = q$ and calculate the quantity $S(k) \equiv S^{\Gamma\backslash\mathcal{H}}(k)(S^{\mathcal{H}}(k))^{-1}$ needed in equations (20)–(23). If we once again apply the functional relation for the Riemann-zeta function with $s = 2\mathrm{i}k$ and $-2\mathrm{i}k$, and the relation $\Gamma(1 \pm \mathrm{i}k) = \pm\Gamma(\mathrm{i}k)$, it is straightforward to prove that

$$S(k) = S^{\Gamma\backslash\mathcal{H}}(k)(S^{\mathcal{H}}(k))^{-1} = -(2\pi)^{-2\mathrm{i}k} \frac{\zeta(2\mathrm{i}k)}{\zeta(-2\mathrm{i}k)}. \tag{52}$$

We can see that the 'renormalized' S-matrix can be written entirely in terms of a fluctuating term, the background term containing the Γ function was eliminated. Our next task is to calculate the renormalized time delay $T(k)$ of (23). Note that the $T^{\Gamma\backslash\mathcal{H}}(k)$ part of the time delay for the leaky torus has already been analysed carefully in the paper of Wardlaw and Jaworski [23], hence we merely reformulate their result in terms of $T(k)$, which is more convenient for our purposes. First, we write $T(k)$ in the form

$$T(k) = \frac{1}{k}\left(\log 2\pi - 2\mathrm{Re}\,\frac{\zeta'}{\zeta}(2\mathrm{i}k)\right). \tag{53}$$

Now we use the formula

$$-\frac{\zeta'}{\zeta}(s) = \frac{s}{s-1} - \sum_{\varrho} \frac{s}{\varrho(s-\varrho)} + \sum_{n=1}^{\infty} \frac{s}{2n(s+2n)} - \frac{\zeta'}{\zeta}(0) \tag{54}$$

with $\frac{\zeta'}{\zeta}(0) = \log 2\pi$ (see pp 52 and 66 of [24]) to write $T(k)$ in terms of the Riemann zeros, $\varrho = \frac{1}{2} - 2\mathrm{i}k_{\varrho}$ of $\zeta(s)$. Since [24]

$$\sum_{\varrho} \frac{1}{\varrho} = \frac{1}{2}\gamma + \frac{1}{2}\log\pi + 1 - \log 2\pi \tag{55}$$

straightforward calculation yields the result

$$\begin{aligned} kT(k) = \log 2 - \gamma + \sum_{n=1}^{\infty} \frac{k^2}{n(n^2+k^2)} \\ -\frac{1/2}{(1/2)^2 + k^2} + \sum_{k_{\varrho}>0}\left(\frac{1/4}{(1/4)^2 + (k+k_{\varrho})^2} + \frac{1/4}{(1/4)^2 + (k-k_{\varrho})^2}\right). \end{aligned} \tag{56}$$

The first three terms correspond to the time delay of $-T^{\mathcal{H}}$ of (29), the fourth term corresponds to the obvious point $s_0 = 1$, $E_0 = 0$ in the residual spectrum. The last term produces the resonances we are interested in. We see that the special form of $T(k)$ fits into the general scheme suggested by equations (29)–(32). Moreover, we see that $\Gamma_\mu = \frac{1}{2}$, and $k_\mu = r_\mu/2$, where r_μ, $\mu = 1, 2, \ldots$, ranges over the zeros of Riemann's zeta function.

Our next example covers the case of calculating the time delay $T(k)$ for a three-channel scattering problem. Let us choose the group $\Gamma \equiv \Gamma(2)$ of (1) as our Fuchsian group. Note that for the principal congruence group $\Gamma(N)$ $(N > 1)$, $g = 1 + \mu_N \frac{N-6}{12N}$ and $\kappa = \mu_N/N$, where $\mu_N = \frac{N^3}{2}\prod_{p|N}(1-p^{-2})$ $N > 2$, $\mu_2 = 6$ is the index of Γ_N in $SL(2,\mathbb{Z})$ [11]. Note that there are no elliptic points for $N > 1$, hence for $N = 2$ we have $(g, \kappa, e) = (0, 3, 0)$. The Riemann surface in question is a sphere with three cusps, i.e. pants. It is interesting to note here that the compact surfaces of type $(g, 0, 0)$ can be decomposed into pants—a fact which is used in string theory. The S-matrix for this surface is straightforward to calculate following the paper of Pnueli [10], the result is

$$S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}(q,k) = -\left(\frac{q}{\pi}\right)^{-2\mathrm{i}k} \frac{\Gamma(\frac{1}{2}-\mathrm{i}k)}{\Gamma(\frac{1}{2}+\mathrm{i}k)} \frac{\zeta(1-2\mathrm{i}k)}{\zeta(1+2\mathrm{i}k)} \mathcal{R}_{\alpha\beta}(k) \qquad \alpha,\beta = 1,2,3 \tag{57}$$

where

$$\mathcal{R}_{\alpha\beta}(k) = \begin{pmatrix} x & y & y \\ y & x & y \\ y & y & x \end{pmatrix} \qquad x = \frac{2^{-4\mathrm{i}k}}{2-2^{-2\mathrm{i}k}} \qquad y = \frac{2^{-2\mathrm{i}k} - 2^{-4\mathrm{i}k}}{2-2^{-2\mathrm{i}k}}. \tag{58}$$

It is easy to check that this matrix is as unitary and symmetric as it has to be. Note that up to the matrix $\mathcal{R}_{\alpha\beta}(k)$ we have the same structure as in our previous example. The 3×3 matrix $\mathcal{R}$ has no effect on the fluctuating part. In order to calculate its contribution to the time delay $T(k)$ we note that $\mathrm{Det}\mathcal{R} = (x-y)^2(x+2y) = 2^{-2\mathrm{i}k}(\frac{2^{1-2\mathrm{i}k}-1}{2^{1+2\mathrm{i}k}-1})$, hence we have

$$\begin{aligned} \frac{\mathrm{i}}{2k}\partial_k \log \mathrm{Det}\mathcal{R}(k) &= \frac{2\log 2}{k}\left(\frac{1}{1-2^{-1+2\mathrm{i}k}} + \frac{1}{1-2^{-1-2\mathrm{i}k}}\right) + 2\log 2 \\ &= \frac{\log 2}{k}\left(\frac{21 - 12\cos(2k\log 2)}{5 - 4\cos(2k\log 2)}\right). \end{aligned} \tag{59}$$

Since we have three cusps the total time delay is this contribution plus three times the one obtained from equation (56).

We cannot resist the temptation to present a calculation of the time delay $T(E)$ for yet another scattering matrix that has appeared in the literature. In this case, however, the scattering systems are slightly different from the ones described so far. The difference manifests itself in the presence of elliptic or orbifold points on the surfaces in question. Although these points render our surface not to be a manifold but rather an orbifold, the scattering contribution to the trace formula is obviously left intact by these points. Indeed [12, 25], the only effect of these points is to change the multiplicity of the *trivial zeros* (i.e. the ones that are at the points $s = -n$, $n = 1, 2, \ldots$) of Selberg's zeta function, and adding a new term to $N_s(E)$ (behaving asymptotically as $\mathrm{O}(1)$) in the functional equation of $Z(s)$.

This class of scattering systems arises by taking the Fuchsian group as $\Gamma_0(N)$, i.e. the Hecke congruence group with a *special choice of* N. Note that $\Gamma_0(N)$ is a subgroup of the modular group containing $\Gamma(N)$ of (1) defined as

$$\Gamma_0(N) \equiv \left\{\gamma \in SL(2,\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\bmod N)\right\}. \tag{60}$$

The values of N we are interested in are square-free integers. These are integers of the form $N = p_1 p_2 \ldots p_r$, where r refers to the number of *distinct* prime factors in the canonical form of N. It can be shown [12] that the number of scattering channels (inequivalent cusps) equals $\kappa = 2^r$. Then, slightly rewriting the result due to Hejhal [12], for $S_{\alpha\beta}^{\Gamma\backslash\mathcal{H}}(q,k)$ we get an expression having the same form as equation (57) with $\alpha, \beta = 1, 2, \ldots, 2^r$, and the matrix $\mathcal{R}_{\alpha\beta}(k)$ now having the form

$$\mathcal{R}(k) = \otimes_{p|N}\mathcal{R}_p(\tfrac{1}{2} + \mathrm{i}k) \qquad \mathcal{R}_p(s) = M_p(1-s)M_p^{-1}(s) \qquad M_p(s) = \begin{pmatrix} 1 & p^s \\ p^s & 1 \end{pmatrix}. \tag{61}$$

A straightforward calculation for the time delay yields the expression 2^r times the one given by (56) and the term

$$\frac{\mathrm{i}}{2k}\partial_k \log \mathrm{Det}\mathcal{R}(k) = \frac{2^r}{k}\sum_{p|N} p \log p \left(\frac{p - \cos(2k\log p)}{1 - 2p\cos(2k\log p) + p^2}\right). \tag{62}$$

For $N = p$ we see that we have two channel scattering problems. It is interesting to examine those choices for the prime p when we have *no elliptic fixpoints*. Hence in this case we have a class of $\Sigma_{g,2}$ surfaces. According to [11], there are two types of elliptic fixpoints, points of order two and three. Let us denote the number of such points by e_2, and e_3. These numbers for square-free N are given by the expressions [11]

$$e_2 = \prod_{p|N}\left(1 + \left(\frac{-1}{p}\right)\right) \qquad e_3 = \prod_{p|N}\left(1 + \left(\frac{-3}{p}\right)\right) \tag{63}$$

where the values of the quadratic residue symbol are

$$\left(\frac{-1}{p}\right) = \begin{cases} 0 & \text{for} \quad p = 2 \\ 1 & \text{for} \quad p \equiv 1 \bmod 4 \\ -1 & \text{for} \quad p \equiv 3 \bmod 4 \end{cases} \tag{64}$$

$$\left(\frac{-3}{p}\right) = \begin{cases} 0 & \text{for} \quad p = 3 \\ 1 & \text{for} \quad p \equiv 1 \bmod 3 \\ -1 & \text{for} \quad p \equiv 2 \bmod 3. \end{cases} \tag{65}$$

For $N = p$ we have merely one term in the product, hence the condition for not having any elliptic point is the simulteneous fulfillment of the linear congruence system $p \equiv 2 \bmod 3$, and $p \equiv 3 \bmod 4$. Since $(3, 4) = 1$ according to the Chinese remainder theorem this system has merely one solution modulo $4 \times 3 = 12$. The least prime solving this system is $p = 11$, hence for the primes having the form $p_n = 11 + 12n$, $n = 0, 1, 2, \ldots$, we have no elliptic points. Moreover [11], for the genus we have the formula $g = 1 + (p+1)/12 - e_2/4 - e_3/3 - \kappa/2$. Inserting here $\kappa = 2$, $e_2 = e_3 = 0$, we get the formula $g_n = (p_n + 1)/12 = n + 1$. Hence for $p_0 = 11$ we get a Riemann surface with genus 1 (i.e. a torus) with two cusps. Since $(11, 12) = 1$ according to the famous theorem of Dirichlet [28], there are an *infinite* number of primes in the arithmetic progression $p_n = 11 + 12n$ and we have an infinite number of such $\Sigma_{g_n,2}$ surfaces. Such surfaces have genus $g = 1, 2, 4, 5, 6, 7, 9, 11, \ldots$. Now we see that the 2×2 scattering matrices can be parametrized instead of such p_n primes, also by the genus g_n, by writing $p_n = 12g_n - 1$ in equation (61). Hence in this case the matrix-valued term in the scattering matrix depends not only on the energy, but also *on the genus* of the surface.

Even though this method presents a multitude of chaotic scattering problems, we see that in all of these examples chaos always manifests itself through the presence of *the same term*, namely the common summand, equation (56). Moreover, we also know that the irregular distribution of scattering resonances can be traced back to the corresponding distribution of the nontrivial zeros of Riemann's zeta function. Indeed, it is not hard to see that the terms reflecting the multichannel nature of the scattering problem produce no new zeros in Selberg's zeta function, hence no new resonances will be found.

Do we always obtain time delays with the same structure for arithmetic subgroups of the modular group? The answer is negative. Even for the group $\Gamma(N)$ (as observed by Pnueli [10]) we obtain expressions with Dirichlet L-series replacing Riemann's zeta function (see also [12]). Unfortunately, we are supplied with a limited number of scattering matrices that were calculated explicitly. For non-arithmetical subgroups, as far as the author knows, no case has ever been analytically solved.

8. Including an integer magnetic field

In this section we examine how the inclusion of an *integer* magnetic field makes its appearance in Selberg's trace Formula. In this case the corresponding quantum systems are the ones arising from the quantization of the classical motion of a particle on $\Sigma_{g,\kappa}$ in a constant magnetic field B. In what follows we first summarize known results that can be found in [5, 10, 12, 14].

When one would like to study electrodynamics on topologically non-trivial configuration spaces $Q = \Sigma_{g,\kappa}$, the field strength F and the vector potential A have to be regarded as the curvature and connection forms of a principal $U(1)$ bundle over Q. F is a two-form which is closed due to Maxwell's equations, however, it cannot always be represented in the form $F = \mathrm{d}A$ *globally*, if $H^2(Q, \mathbb{R})$ is nontrivial. However, in spite of $H^2(Q, \mathbb{R})$ being nontrivial we still have the chance to define elctrodynamics on Q provided $[\frac{1}{2\pi}F] \in H^2(Q, \mathbb{Z})$ i.e., if F suitably normalized defines an integral second cohomology class. More precisely, there exists a complex line bundle L over Q with connection ∇, with F being its curvature form iff $[\frac{1}{2\pi}F] \in H^2(Q, \mathbb{Z})$. Then in the local coordinates x^a, $a = 1, 2$, the connection reads as $\nabla_a = \partial_a - \mathrm{i}A_a$, with A_a a local vector potential in a given gauge. The Hamiltonian is $H = -\nabla^2$, and the wavefunctions of the system are (local) sections of L.

Making use of Riemann uniformization, we again represent our $Q = \Sigma_{g,\kappa}$ as $\Gamma\backslash\mathcal{H}$. With the help of the Poincaré metric on $\mathcal{H}$ we can define an orientation and the volume form $y^{-2}\,\mathrm{d}x \wedge \mathrm{d}y$ on $\Sigma_{g,\kappa}$. (The volume form is invariant with respect to the action of Γ.) A constant magnetic field is defined by demanding that F be proportional the volume form, i.e. $F = By^{-2}\,\mathrm{d}x \wedge \mathrm{d}y$. According to the Gauss–Bonnet theorem $\Sigma_{g,\kappa}$ has finite volume $V = 2\pi(2g + \kappa - 2)$, hence combining this with the fact that $[\frac{1}{2\pi}F] \in H^2(Q, \mathbb{Z})$ for the flux we obtain $\int F = BV = 2\pi\mathbb{Z}$, i.e. the flux on $\Sigma_{g,\kappa}$ has to be quantized. Moreover, we see that B must be rational. However, further arguments [5, 14] show that a non-integer B can only be introduced consistently provided we also introduce fluxes through the cusps. Hence in the absence of fluxes we have $B \in \mathbb{Z}$. In the following we suppose that no additional fluxes are present, hence $B \in \mathbb{Z}$ and moreover, we take $B \geqslant 2$.

Choosing the gauge $A = y^{-1}B\,\mathrm{d}x$, our physical system is governed by the Hamiltonian $H(B) = -\nabla^2 = -y^2(\partial_x^2 + \partial_y^2) + 2\mathrm{i}By\partial_x + B^2$, subject to the boundary condition [5, 10, 14] $\psi(\gamma z) = \left(\frac{cz+d}{|cz+d|}\right)^{2B}\psi(z)$, where $\gamma \in \Gamma$.

Now we are ready to present Selberg's trace formula for the Hamiltonian $H(B)$, with B a constant magnetic field. Let $\lambda_j = \frac{1}{4} + B^2 + {k_j}^2$ denote an eigenvalue belonging to the discrete part of the spectrum, i.e. we have $H(B)\psi_j = \lambda_j\psi_j$. Moreover, let $C = \max[B - \frac{1}{2}, \frac{1}{2}]$ and $\kappa \geqslant 1$, $B > 0$, and $h(k)$ be a function satisfying the following conditions:

$$\begin{array}{ll} h(k) & \text{is even} \\ h(k) & \text{holomorphic in the strip} \quad |\mathrm{Im}\,k| \leqslant C + \varepsilon \\ h(k) \ll (|k| + 1)^{-2-\varepsilon} & \text{in the strip.} \end{array} \qquad (66)$$

As usual, let $g(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}uk} h(k)\,\mathrm{d}k$ be the Fourier transform of $h(k)$. Then Selberg's trace formula for noncompact surfaces without elliptic points in the presence of a constant magnetic field B is [12]

$$\begin{aligned}\sum_j h(k_j) = {}& \frac{V}{4\pi}\int_{-\infty}^{+\infty} kh(k)\frac{\sinh 2\pi k}{\cosh 2\pi k + \cos 2\pi B}\,\mathrm{d}k + \sum_{\mathrm{PPO}}\sum_{n=1}^{\infty}\frac{l(p)}{2\sinh(nl(p)/2)}g(nl(p)) \\ &+\frac{V}{4\pi}\sum_{1\leqslant n<2B, n\,\mathrm{odd}}(2B-n)h\left(\mathrm{i}\left(B-\frac{n}{2}\right)\right) \\ &+\kappa\int_0^{\infty}\frac{g(u)}{\mathrm{e}^{u/2}-\mathrm{e}^{-u/2}}(1-\cosh Bu)\,\mathrm{d}u \\ &+\frac{1}{4}h(0)\,\mathrm{Tr}\left(I-\Phi\left(B,\frac{1}{2}\right)\right) - \kappa g(0)\log 2 - \frac{\kappa}{2\pi}\int_{-\infty}^{\infty} h(k)\psi(1+\mathrm{i}k)\,\mathrm{d}k \\ &+\frac{1}{4\pi}\int_{-\infty}^{\infty} h(k)\,\mathrm{Tr}\left(\Phi'\left(B,\frac{1}{2}+\mathrm{i}k\right)\Phi\left(B,\frac{1}{2}+\mathrm{i}k\right)^{-1}\right)\mathrm{d}k \end{aligned} \tag{67}$$

where the $\kappa\times\kappa$ matrix $\Phi(B,\frac{1}{2}+\mathrm{i}k)$ proportional to the the scattering matrix in the presence of a constant integer magnetic field is [10]

$$\Phi(B, 1/2+\mathrm{i}k) = (-1)^B\frac{\Gamma(1/2+\mathrm{i}k)^2}{\Gamma(1/2+B+\mathrm{i}k)\Gamma(1/2-B+\mathrm{i}k)}\Phi(1/2+\mathrm{i}k) \tag{68}$$

where $\Phi(\frac{1}{2}+\mathrm{i}k)$ is proportional to the scattering matrix without magnetic field defined by equations (4) and (8).

Now we would like to consider the B-dependent terms in the trace formula for the B *integer*. First we examine the term $I_1(B) = -\frac{\mathrm{i}}{4\pi}\int_{-\infty}^{\infty} h(k)\partial_k \log\mathrm{Det}(\Phi(B,\frac{1}{2}+\mathrm{i}k))\,\mathrm{d}k$. We easily deduce that $\mathrm{Det}\Phi(B+1,\frac{1}{2}+\mathrm{i}k) = \left(\frac{1/2+B-\mathrm{i}k}{1/2+B-f\mathrm{i}k}\right)^{\kappa}\mathrm{Det}\Phi(B,\frac{1}{2}+\mathrm{i}k)$. Hence

$$I_1(B) - I_1(B-1) = -\frac{\kappa}{2\pi}\int_0^{\infty} h(k)\frac{2B-1}{(B-\frac{1}{2})^2+k^2}. \tag{69}$$

Since $h(k)$ is an even function, in the last integral we can substitute the expression $h(k) = 2\int_0^{\infty}\mathrm{d}u g(u)\cos ku$ and use the integral

$$\int_0^{\infty}\mathrm{d}k\cos ku\frac{1}{(B-1/2)^2+k^2} = \frac{\pi}{2B-1}\mathrm{e}^{-(B-1/2)u} \qquad B-\tfrac{1}{2}>0 \tag{70}$$

to arrive at the result

$$I_1(B) - I_1(B-1) = -\kappa\int_0^{\infty}\mathrm{d}u g(u)\mathrm{e}^{-(B-1/2)u}. \tag{71}$$

Now we examine the other term $I_2(B) \equiv \kappa\int_0^{\infty}\frac{g(u)}{\mathrm{e}^{u/2}-\mathrm{e}^{-u/2}}(1-\cosh Bu)\,\mathrm{d}u$ modifying the parabolic contribution when a nonzero B field is present. First we note that

$$\cosh Bu - \cosh(B-1)u = \tfrac{1}{2}(\mathrm{e}^{u/2}-\mathrm{e}^{-u/2})(\mathrm{e}^{(B-1/2)u}-\mathrm{e}^{-(B-1/2)u}). \tag{72}$$

Using this and the definition of $g(u)$ for the quantity $I_2(B) - I_2(B-1)$ we get the expression

$$I_2(B) - I_2(B-1) = \frac{\kappa}{2}\int_0^{\infty}\mathrm{d}u g(u)(\mathrm{e}^{-(B-1/2)u}-\mathrm{e}^{(B-1/2)u}). \tag{73}$$

For $J(B) \equiv I_1(B) + I_2(B)$ (the B-dependent terms in the parabolic contribution) we get the formula

$$J(B) - J(B-1) = -\kappa\int_0^{\infty}\mathrm{d}u g(u)\cos\mathrm{i}\left(B-\frac{1}{2}\right)u = -\frac{\kappa}{2}h\left(\mathrm{i}\left(B-\frac{1}{2}\right)\right). \tag{74}$$

Applying this formula successively we obtain

$$J(B) = J(0) - \frac{\kappa}{2} \sum_{1 \leqslant n < 2B, n \text{ odd}} h\left(\mathrm{i}\left(B - \frac{n}{2}\right)\right). \tag{75}$$

Hence for *integer* magnetic field the B-dependence manifests itself in the parabolic terms merely in the form of the sum appearing in equation (75).

The first term on the right-hand side of the trace formula for *integer* B reduces to the well known term $\frac{A}{4\pi}\int_{-\infty}^{+\infty} kh(k)\tanh(\pi k)\,\mathrm{d}k$. Introducing the numbers for $B \geqslant 2$

$$D_m(B) \equiv \frac{V}{4\pi}(2B - 2m - 1) - \frac{\kappa}{2} = (2B - 2m - 1)(g - 1) + (B - m - 1)\kappa \qquad m = 0, 1, \ldots \tag{76}$$

and combining the third term on the right-hand side of trace formula (67) with the term containing the sum in (75), we get the sum

$$\sum_{0 \leqslant m < B - 1/2} D_m(B) h(\mathrm{i}(B - m - \tfrac{1}{2})) = \sum_{0 \leqslant m < B - 1/2} D_m(B) h(k_m) \tag{77}$$

where using $k_m = \mathrm{i}(B - m - \frac{1}{2})$ in the formula $E_m = \frac{1}{4} + B^2 + k_m{}^2$ we get $E_m = (2m + 1)B - m(m + 1)$, $m = 0, 1, \ldots, B - 1$. Note that $E_0 = B$ and $E_{B-1} = B^2$, hence $E_m \in [B, B^2]$. This part of the spectrum is reminiscent of the usual Landau levels, moreover, the numbers $D_m(B)$ are the degeneracies of these Landau levels [26, 27].

According to equation (75) for the integer magnetic field the dependence on B can be entirely transferred to the term containing the sum over discrete Landau levels. Hence the integer magnetic field does not effect the scattering states in the trace formula. Moreover, according to [5] the second term on the right-hand side of the trace formula (67) (the periodic orbit sum) describing the 'chaotic' part of the spectrum when an integer magnetic field is present, can be mapped to the corresponding periodic sum *without a magnetic field*. More precisely, there is a one-to-one mapping between the classical periodic orbits with and without a magnetic field. This reflects the fact that the 'chaotic' part of the spectrum with a B field can be obtained from the free one by a constant shift of B^2. Collecting everything for the final form of the trace formula for a particle moving in an integer magnetic field on $\Sigma_{g,\kappa}$ we get

$$\sum_j h(k_j) = \frac{V}{4\pi}\int_{-\infty}^{+\infty} kh(k)\tanh(\pi k)\,\mathrm{d}k + \sum_{0 \leqslant m < B - 1/2} D_m(B) h\left(\mathrm{i}\left(B - m - \frac{1}{2}\right)\right)$$
$$+ \sum_{\mathrm{PPO}} \sum_{n=1}^{\infty} \frac{l(p)}{2\sinh(nl(p)/2)} g(nl(p)) + \frac{\kappa}{2} h(0) - \frac{1}{2\pi}\int_0^{\infty} h(E) T(E)\,\mathrm{d}E \tag{78}$$

where $T(E)$ is the renormalized time delay introduced in equations (22) and (23). Note that $T(E)$ is not dependent on B. Indeed, the only dependence in the trace formula (apart from the aforementioned constant shift in the 'chaotic' part of the spectrum), comes from the second term reflecting the appearence of the Landau levels with appropriate degeneracies.

9. Conclusions

In this paper we considered the problem of quantization of the motion of a charged particle on a surface of constant negative curvature, with or without the influence of a constant magnetic field. The classical motion on such surfaces is known to be strongly chaotic. We have chosen a special class of such surfaces, with points infinitely far away with respect to the metric on them. Such points describe scattering channels. A particle can enter through some of these channels and leave on any of them after some time. A natural physical quantity measuring the time spent during the scattering process is the Wigner–Smith time delay. For usual scattering systems where the interaction is defined by a potential, the time delay is defined as the difference between the time spent by the scattered particle within the region of the potential, and the time that it would have spent in the same region had it moved without the influence of the potential. However, the definition of the time delay is an intricate one when the interaction is merely defined by enforcing special boundary conditions on the wavefunction. In this paper we suggested to identify the free dynamics as the motion on the entire Poincaré upper half-plane $\mathcal{H}$, and the interacting dynamics as restricting this motion to some fundamental domain of $\mathcal{H}$ tessellating it. This domain is defined by a discrete subgroup Γ of the isometry group of $\mathcal{H}$. The special form of boundary conditions reflects the special geometry of the surfaces. Using Selberg's trace formula for noncompact Riemann surfaces, we showed that the natural quantity for the time delay associated with such 'geometric' scattering is the *renormalized time delay* which is the time delay associated with the surface in question *minus* the time delay corresponding to the scattering problem on the Poincaré upper half-plane uniformizing our surface. Based on known results we clarified the pole structure of the scattering matrix, and examined how these poles manifest themselves in the trace formula. The physical meaning of these poles is clarified after introducing Selberg's zeta function which is described by purely *classical* data, i.e. the length spectra of classical periodic orbits. It was shown that the a part of the nontrivial zeros of Selberg's zeta function can be related to the presence of poles in the S-matrix of the corresponding *quantal* problem. These poles describe scattering resonances. It was also shown that the inclusion of an integer $B \geqslant 2$ magnetic field has no effect on the features of these resonances. Our results were illustrated for the surfaces $\Sigma_{1,1}$ (Gutzwiller's leaky torus), $\Sigma_{0,3}$ (pants), and a class of $\Sigma_{g,2}$ surfaces.

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References

[1] Balázs N L and Voros A 1986 *Phys. Rev.* **143** 109
[2] Gutzwiller M C 1971 *J. Math. Phys.* **12** 343
[3] Selberg A 1956 *J. Ind. Math. Soc.* **20** 47
[4] Matthies C and Steiner F 1991 *Phys. Rev.* A **44** R7877
Aurich R, Seiber M and Steiner F 1988 *Phys. Rev. Lett.* **61** 483
[5] Comtet A, Georgeot B and Ouvry S 1993 *Phys. Rev. Lett.* **71** 3786
[6] Faddeev L D 1967 *Trans. Moscow Math. Soc.* **17** 357
[7] Lax P D and Phillips R S 1976 *Scattering Theory for Automorphic Functions* (Princeton NJ: Princeton University Press)
[8] Gutzwiller M 1983 *Physica* D **7** 341
[9] Gutzwiller M 1990 *Chaos in Classical and Quantum Mechanics* (Berlin: Springer)
[10] Pnueli A 1994 *Ann. Phys., NY* **231** 56
[11] Shimura G 1971 *Introduction to the Arithmetic Theory of Automorphic Functions* (Princeton, NJ: Princeton University Press)
[12] Hejhal D 1983 *The Selberg Trace Formula for $PSL(2,\mathbb{R})$ vol 2 (Lecture Notes in Mathematics vol 1001)* (Berlin: Springer)
[13] Kubota T 1975 *Elementary Theory of Eisenstein Series* (New York: Wiley)
[14] Antoine M, Comtet A and Ouvry S 1990 *J. Phys. A: Math. Gen.* **23** 3699
[15] Apagyi B, Endrédi G and Lévay P (ed) 1997 Inverse and algebraic quantum scattering theory *Proc. Conf. (Lake Balaton, Hungary, Sept 1996)* (Berlin: Springer)
[16] Werhahn R F 1990 *Phys. Rev. Lett.* **65** 1294

[17] Kerimov G A and Sezgin M 1998 *J. Phys. A: Math. Gen.* **31** 7901
[18] Abramowitz M and Stegun I A 1968 *Handbook of Mathematical Functions* (New York: Dover)
[19] Wigner E P 1955 *Phys. Rev.* **98** 145
Smith F 1960 *Phys. Rev.* **118** 349
[20] McKean H P 1972 *Commun. Pure Appl. Math.* **25** 225
[21] Iwaniec H 1995 *Introduction to the Spectral Theory of Automorphic Forms Bibl. Rev. Mat. Iber.* (Madrid: Alhambra)
[22] Fyodorov Y V and Sommers H 1997 *J. Math. Phys.* **38** 1918
[23] Wardlaw D M and Jaworski W 1989 *J. Phys. A: Math. Gen.* **22** 3561
[24] Edwards H M 1974 *Riemann's Zeta Function* (New York: Academic)
[25] Venkov A B 1981 Spectral theory of automorphic functions *Proc. Steklov Inst. Math.* **153** (*Am. Math. Soc.)* Transl.
[26] Avron J E, Klein M, Pnueli A and Sadun L 1992 *Phys. Rev. Lett.* **69** 128
[27] Comtet A 1987 *Ann. Phys., NY* **173** 185
Grosche C 1988 *Ann. Phys., NY* **187** 110
[28] Apostol T 1976 *Introduction to Analytic Number Theory* (New York: Springer)